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Asymptotic model of the mobile interface between two liquids in a thin porous stratum

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We propose a new generalized model to describe deformations of the mobile interface separating two immiscible liquids in a porous medium. The densities and the viscosities of the fluids can have any value. The horizontal size of the interface is much greater than the vertical size of the domain. Unlike the classical theory, the new model describes gravitational non-equilibrium processes, including the Rayleigh–Taylor instability which appears in the form of a negative apparent diffusion parameter. Several flow regimes are established depending on the ratio between gravity and the elastic fluid/medium forces, and between the vertical and horizontal flow rates. The model is used to analyse the evolution of the interface during the free spreading of one liquid over another. This is characterized by the presence of interface degeneration points. The explicit solution to the problem of oil and water flow towards a well is presented as an application to oil reservoirs.

1. Introduction

A pair of immiscible liquids in porous medium is examined. They are assumed to be separated by a mobile interface which is horizontal in the initial state. The liquids have different viscosity and density. The vertical size of the domain is assumed to be small with respect to the horizontal size of the interface (shallow-water model).

Two-phase flow with a mobile interface, which is one of the more complicated subjects of mathematical analysis, is usually described by a system of partial differential equations which are valid on either side of the interface, while at the interface they are bound by some dynamic and kinetic conditions. One of the most important scientific problems consists in converting such a system into an explicit closed differential equation for a selected coordinate of the mobile surface. More generally, if the interface equation is $F(x_1, x_2, z, t) = 0$, or $z = h(x_1, x_2, t)$, then the problem is to deduce the closed differential equation for the function $F(x_1, x_2, z, t)$ or $h(x_1, x_2, t)$.

In fluid mechanics this can be done for three basic cases. First, there is the case where the viscous forces can be neglected for both fluids (Whitham 1974), which does not occur in flow through porous media. Secondly, there is the case where one of two liquids is not viscous, as for instance in the case of water–air flow. The shallow-water theory (Whitham 1974) yields a classical example of such an explicit model for the function h describing the surface waves on water. The model for h takes the form of the Korteweg/deVries equation. Another example is the groundwater flow in an unconfined aquifer (Bear 1972; Barenblatt, Entov & Ryzhik 1990) which leads to the nonlinear parabolic Boussinesq equation with respect to h. In Barenblatt *et al.* (1990) this model has been deduced using an integration of flow equations over the vertical coordinate z and assuming a hydrostatic pressure distribution along z. In Liu & Wen (1997) the same model has been obtained using the asymptotic expansion method. The vertical size of the porous reservoir is assumed to be much smaller than its horizontal length. More general models have been obtained in Dagan (1967) and Parlange *et al.* (1984) where the gravitational equilibrium assumption has been removed.

When both fluids are viscous (the third case), the approximate, explicit equations were deduced in Polubarinova-Kochina (1962), assuming both the horizontal flow velocity to be constant along z and a steady-state flow for both liquids. The first condition is often replaced by the gravitational equilibrium condition.

In the case of a two-liquid flow, the condition of gravitational equilibrium, i.e. a hydrostatic pressure distribution, becomes excessively restrictive. It does not enable a description of the development of fast gravitational disturbances. In particular, it is impossible to analyse the evolution of instability within the framework of such a model. The condition of a steady-state flow does not enable the flow of compressible liquids to be studied, when the elastic disturbance is spreading more slowly than the gravitational disturbance.

In the initial attempts to obtain a model for two fluids in a thin reservoir (Panfilov, Crolet & Calugaru 2000, 2001), some strong simplifying assumptions were introduced. In particular, the interface deformations were considered small. The lateral (horizontal) flow velocity was totally neglected and a partial linearization was performed. As a result, the equations obtained described only a single particular case of non-lateral flow and, moreover, in a simplified manner. Nevertheless, the effect of a connection between gravitational instability and the anti-diffusion equation was noted.

In this paper both fluids are assumed to be viscous and compressible, the hypothesis of gravitational equilibrium is rejected, the flow is non-stationary and no linearization has been performed. The simplified assumptions concerning minor deformations of the interface are not used. The lateral flow may now play a dominant role. It may exhibit any value between zero and infinity, and, as will be shown later, determines several flow regimes and a qualitative physical effect of suppressing Rayleigh–Taylor instability. To obtain the model we have used the integral relations method, similar to that developed by Kármán–Polhausen in boundary layer theory (Darrozes & François 1998). It consists of integrating the flow equations over the height of each liquid layer and closing the integral relations obtained through a hypothesis concerning the vertical velocity distribution along the vertical coordinate. In the case of the thin stratum under study, this method is equivalent to the asymptotic expansion technique.

2. Formulation of the problem

2.1. *Physical model*

Let us introduce an orthogonal coordinate system (x_1, x_2, z), where z is a selected 'vertical' coordinate, which coincides with the gravity vector, and where (x_1, x_2) are coordinates of the horizontal plane, which may not necessarily be Cartesian.

Let us examine a horizontal porous stratum of height $H(x_1, x_2)$, which occupies the domain $\Omega \subset R^3$. Here, two immiscible liquids are separated from one another by an interface, which does not cross the top and bottom stratum boundaries, as shown

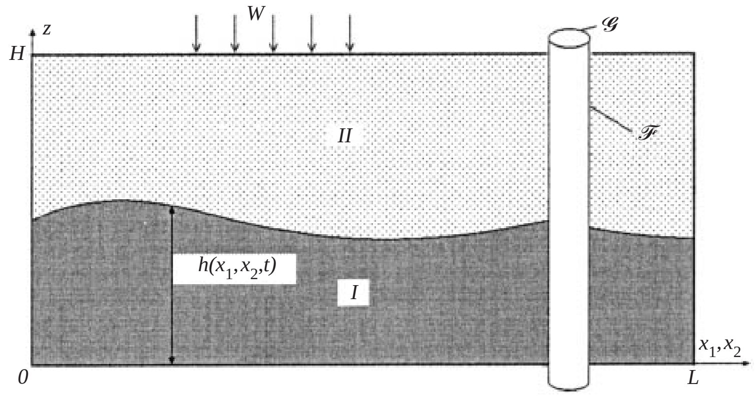


FIGURE 1. Diagram of the process.

in figure 1. The lateral limits of the domain do not play an important role, i.e. the vertical size H is much smaller than the horizontal length L. In particular, the domain can be considered as infinite along x_1, x_2.

Let $h(x_1, x_2, t)$ denote the height of the interface with respect to the bottom of the stratum. The indexes $i = I, II$ correspond respectively to the lower and the upper liquids. Let the bottom boundary of the domain be impermeable, whereas the top surface is open to a flow across it. The normal velocity, W, of the crossflow is assumed to be known at each point of the top boundary. The lateral, i.e. horizontal, flow is maintained by a characteristic pressure difference, ΔP, imposed at the lateral boundaries of the domain. The porous stratum is assumed to be homogeneous but anisotropic with a diagonal permeability tensor $\boldsymbol{K} \equiv \{K_{ij}\}_{i,j=1}^{3}$, such that: $K_z \equiv K_{33}$, $K_{x_1} \equiv K_{11}$, $K_{x_2} \equiv K_{22}$, and $K_{ij} = 0$, $i \neq j$. Within the framework of the present paper we assume $K_{x_1} = K_{x_2} \equiv K_x$, so only the difference between the vertical (K_z) and the horizontal permeability (K_x) is of interest. The capillary forces are neglected. A special cylindrical surface $\mathscr{F}$ representing a well may penetrate the stratum as shown in figure 1.

2.2. *Flow equations*

The flow of the slightly compressible ith liquid in a weakly elastic medium can be described by the usual system of equations with respect to the liquid pressure P^i and flow velocity $\boldsymbol{V}^i$:

$$\phi\beta_* \frac{\partial P^i}{\partial t} + \operatorname{div} \boldsymbol{V}^i = 0, \quad \boldsymbol{V}^i = -\frac{1}{\mu^i}\boldsymbol{K} \otimes \operatorname{grad} \Phi^i, \quad \Phi^i = P^i + \varrho^i g z, \qquad i = I, II,$$

where ϱ is fluid density, g is acceleration due to gravity, μ is the fluid dynamic viscosity, ϕ is porosity, and $1/\beta_*$ is the measure of fluid/medium compressibility (the smaller $1/\beta_*$, the more compressible the system). Symbol $\otimes$ denotes the tensor product.

This system can be converted into two equations written with respect to functions Φ^I and Φ^{II}:

$$\frac{\partial \Phi^i}{\partial t} = \frac{\partial}{\partial x_k}\left(æ_{kj}\frac{\partial \Phi^i}{\partial x_j}\right), \quad i = I, II. \tag{2.1}$$

Herein the summation is with respect to repeated indexes k and j; the tensor $æ^i_{kj}$ is defined as $æ^i_{kj} = K_{kj}/(\beta_* \mu^i \phi)$.

2.3. *Kinematic equation for the interface*

To proceed further, an explicit kinematic equation for the height of the mobile interface $h(x_1, x_2, t)$ can be written in the following way. Let the interface equation be

$$z = h(x, t), \quad x = \{x_1, x_2\}. \tag{2.2}$$

Function (2.2) is assumed to be existent and unique, therefore the formation of loops on the interface is excluded.

Assuming that the interface remains smooth at all points, we obtain the following kinematic equation:

$$\frac{\partial h}{\partial t} + \boldsymbol{U}(h) \cdot \mathrm{grad} h = U_z(h), \tag{2.3}$$

with $\boldsymbol{U}$ the real surface velocity.

2.4. *Conditions at the interface*

The following three necessary conditions should be imposed on the interface: (i) continuity of pressure; (ii) continuity of normal flow velocity; (iii) the velocity of the interface should be equivalent to the physical velocity of the liquid at each point of the interface, expressed in the following form:

$$P^I\big|_h = P^{II}\big|_h \equiv P(h), \tag{2.4a}$$

$$V_n^I\Big|_h = V_n^{II}\Big|_h, \tag{2.4b}$$

$$\phi\frac{\partial h}{\partial t} + \boldsymbol{V}^I(h) \cdot \mathrm{grad} h = V_z^I(h), \tag{2.4c}$$

$$\phi\frac{\partial h}{\partial t} + \boldsymbol{V}^{II}(h) \cdot \mathrm{grad} h = V_z^{II}(h), \tag{2.4d}$$

because the real flow velocity is equal to $\boldsymbol{V}/\phi$.

Only two of the above four relations are independent. Below, we will use equation (2.4c) and the following, resulting from (2.4c) and (2.4d):

$$\boldsymbol{V}^I(h)\mathrm{grad} h - V_z^I(h) = \boldsymbol{V}^{II}(h)\mathrm{grad} h - V_z^{II}(h). \tag{2.5}$$

2.5. *Boundary-value and initial conditions*

Both the top and the bottom surfaces of the stratum are horizontal. The following conditions express the zero flow across the bottom surface and a fixed flow velocity W across the top surface:

$$V_z^{II}\Big|_{z=H} = -W(x_1, x_2, t), \quad V_z^I\Big|_{z=0} = 0. \tag{2.6}$$

The positive direction of vector W means inflow into the stratum.

At the lateral boundary of the domain the characteristic pressure drop ΔP is defined, for instance, as the difference between the minimal and the maximal mean (i.e. averaged over the stratum thickness) pressure values at the lateral surface of the domain.

The initial conditions define an undisturbed state:

$$h\big|_{t=0} = h_0, \quad P\big|_{t=0,\ z=h_0} = P_0, \quad \boldsymbol{V}^I\big|_{t=0} = \boldsymbol{V}^{II}\big|_{t=0} = 0, \tag{2.7}$$

where h_0 and P_0 are constant.

3. Averaged equations

3.1. *Method of integral relations*

The system of equations with a mobile boundary is difficult to study directly. A more effective way is to simplify the system by integrating the equations along the selected coordinate z and to derive an explicit equation for the interface height $h(x, t)$. However, integration leads to a loss of information about the system behaviour along z. Such a loss must be restored by introducing a closure hypothesis concerning the vertical distribution of the velocity or the pressure field. This technique is a version of the Kármán–Polhausen method of integral relations developed in boundary layer theory (Darrozes & François 1998). Note that Polhausen suggested closure relations in polynomial form. This approach is widely used in shallow water theory.

The closure hypothesis determines to what extent the method is approximate. However, in the case of a thin stratum the integral method is almost insensitive to the closure hypothesis. Indeed, it is clear that, for a thin stratum, the pressure and flow velocity at any point in the vertical section are very close to their mean integral values. Thus in the limit case of an infinitely thin stratum, the integral method converges toward the exact equations whatever the closure hypothesis. The role of the closure hypothesis becomes important only from the first approximation (with respect to the small stratum height H/L). We will choose the closure relation in such a form that in the first approximation it will be correct. Thus in the case of the thin stratum we are studying, the integral method yields the exact results up to and including the first approximation. From this point of view, the integral method, although equivalent to the asymptotic expansion technique, is more elegant.

An asymptotic technique corresponding to a similar thin stratum case was used in Yortsos (1995) for a problem of mixed but non-stratified two-phase flow (described in terms of the saturation but not of the interface). The objective of that paper was to discover the conditions ensuring gravitational equilibrium in the system. In our case, for a stratified fluid we constructed a model which would be valid without the vertical equilibrium condition.

3.2. *Deduction of equations averaged over the vertical coordinate*

Let us perform an integration of (2.1) over intervals $0 \leqslant z \leqslant h$ and $h \leqslant z \leqslant H$, taking into consideration that

$$\begin{aligned}
\int_0^h \frac{\partial \Phi^I}{\partial t}\,\mathrm{d}z &= \frac{\partial}{\partial t}\left(\int_0^h \Phi^I\,\mathrm{d}z\right) - \Phi^I(h)\frac{\partial h}{\partial t} \\
&= \frac{\partial}{\partial t}(h\overline{\Phi}^I) - \frac{\partial}{\partial t}(h\Phi^I(h)) + h\frac{\partial \Phi^I(h)}{\partial t} \\
&= \frac{\partial R^I}{\partial t} + h\frac{\partial \Phi^I(h)}{\partial t},
\end{aligned}$$

$$\begin{aligned}
\int_0^h \frac{\partial^2 \Phi^I}{\partial x_j \partial x_j}\,\mathrm{d}z &= \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}\left(\int_0^h \Phi^I\,\mathrm{d}z\right) - \Phi^I(h)\frac{\partial h}{\partial x_j}\right] - \left.\frac{\partial \Phi^I}{\partial x_j}\right|_h \frac{\partial h}{\partial x_j} \\
&= \frac{\partial^2 R^I}{\partial x_j \partial x_j} + \frac{\partial}{\partial x_j}\left(h\frac{\partial \Phi^I(h)}{\partial x_j}\right) - \left.\frac{\partial \Phi^I}{\partial x_j}\right|_h \frac{\partial h}{\partial x_j},
\end{aligned}$$

etc., where

$$R^I \equiv h(\overline{\Phi}^I - \Phi^I(h)), \quad R^{II} \equiv h^{II}(\overline{\Phi}^{II} - \Phi^{II}(h)), \quad h^{II} \equiv H - h,$$
$$\overline{\Phi}^I \equiv \frac{1}{h}\int_0^h \Phi^I \mathrm{d}z, \quad \overline{\Phi}^{II} \equiv \frac{1}{h^{II}}\int_h^H \Phi^{II}\mathrm{d}z, \quad \Phi^i(h) \equiv \Phi^i\big|_{z=h}. \tag{3.1}$$

The integration yields the set of equations

$$\left.\begin{aligned} \frac{\partial R^I}{\partial t} + h\frac{\partial \Phi^I(h)}{\partial t} &= \mathrm{div}(æ_x^I \mathrm{grad} R^I) + \mathrm{div}(æ_x^I h \mathrm{grad}\Phi^I(h)) - æ_x^I S^I, \\ \frac{\partial R^{II}}{\partial t} + h^{II}\frac{\partial \Phi^{II}(h)}{\partial t} &= \mathrm{div}(æ_x^{II}\mathrm{grad}R^{II}) \\ &\quad + \mathrm{div}(æ_x^{II} h^{II}\mathrm{grad}\Phi^{II}(h)) + æ_x^{II} S^{II} - \frac{æ_x^{II}\mu^{II}\phi W(x)}{K_x}, \end{aligned}\right\} \tag{3.2a}$$

which is completed by conditions at the interface

$$\phi\frac{\partial h}{\partial t} - \frac{K_x}{\mu^I}S^I = 0, \quad \frac{1}{\mu^I}S^I = \frac{1}{\mu^{II}}S^{II}, \tag{3.2b}$$

where

$$S^I \equiv \left.\frac{\partial \Phi^I}{\partial x_j}\right|_h \frac{\partial h}{\partial x_j} - \left.\frac{K_z}{K_x}\frac{\partial \Phi^I}{\partial z}\right|_h, \quad S^{II} \equiv \left.\frac{\partial \Phi^{II}}{\partial x_j}\right|_h \frac{\partial h}{\partial x_j} - \left.\frac{K_z}{K_x}\frac{\partial \Phi^{II}}{\partial z}\right|_h. \tag{3.2c}$$

Relation (3.2) can be simplified by eliminating S^i:

$$\left.\begin{aligned} \frac{\partial R^I}{\partial t} + h\frac{\partial \Phi^I(h)}{\partial t} &= æ_x^I\left[\Delta R^I + h\Delta\Phi^I(h) + \mathrm{grad}h\cdot\mathrm{grad}\Phi^I(h) - \frac{\phi\mu^I}{K_x}\frac{\partial h}{\partial t}\right], \\ \frac{\partial R^{II}}{\partial t} + h^{II}\frac{\partial \Phi^{II}(h)}{\partial t} &= æ_x^{II}\left[\Delta R^{II} + h^{II}\Delta\Phi^{II}(h) - \mathrm{grad}h\cdot\mathrm{grad}\Phi^{II}(h)\right. \\ &\quad \left. + \frac{\phi\mu^{II}}{K_x}\frac{\partial h}{\partial t} - \frac{\mu^{II}\phi W(x)}{K_x}\right], \\ h^{II} \equiv H - h, \quad \Phi^{II}(h) &= \Phi^I(h) + (\varrho^{II} - \varrho^I)gh. \end{aligned}\right\} \tag{3.3}$$

System (3.3) of four equations is not closed as it contains six functions: R^I, R^{II}, $\Phi^I(h)$, $\Phi^{II}(h)$, h and h^{II}.

3.3. *Closure relation: assumptions concerning vertical velocity distribution*

Let us assume the vertical distribution of the vertical flow velocity to be linear:

$$V_z^I(x,z,t) = \frac{K_z a^I(x,t)}{\mu^I}z, \quad V_z^{II}(x,z,t) = \frac{K_z a^{II}(x,t)}{\mu^{II}}(z-H) - W(x,t),$$

where conditions (2.6) have been taken into account; $x \equiv (x_1, x_2)$. New parameters a^I and a^{II} are not yet defined.

Usually the assumption of a hydrostatic pressure distribution is examined (Bear 1972; Barenblatt *et al.* 1990). Such an assumption is equivalent to zero vertical flow velocity and appears satisfactory when the interface deformation is relatively small, when the boundary is free (the upper fluid has no viscosity nor density) and when the transition phenomena are not yet established in the system.

For the more general case studied in this paper, a more general hypothesis is required. Moreover, one can show that the assumption of constant (i.e. zero) vertical flow velocity leads to an overdetermined, contradictory system of equations, as not all the boundary conditions will be satisfied. On the other hand, any other law of velocity distribution (quadratic, etc.) leads to a non-closed system if a single integral relation is used.

In the case of a thin stratum, the assumption of a linear vertical distribution of the vertical velocity appears to be correct. This can be strictly proved using the asymptotic expansion technique with respect to the dimensionless stratum thickness ε which is small. In the Appendix we show that the vertical component of flow velocity is linear along z in the first approximation. Hence the model we present in this paper holds up to and including the terms of $O(\varepsilon)$.

Parameters a^I and a^{II} may be determined from the conditions at the interface, (2.4) or (3.2b):

$$\Phi^I = \Phi^I(h) - \frac{a^I(z^2-h^2)}{2},$$
$$\Phi^{II} = \Phi^{II}(h) - \frac{a^{II}}{2}\left[(z-H)^2-(h-H)^2\right] + \frac{\mu^{II}W}{K_z}(z-h), \tag{3.4}$$

$$\left.\frac{\partial\Phi^I}{\partial z}\right|_h = -ha^I, \quad \left.\frac{\partial\Phi^I}{\partial x_j}\right|_h = \frac{\partial\Phi^I(h)}{\partial x_j} + a^I h\frac{\partial h}{\partial x_j},$$
$$\left.\frac{\partial\Phi^{II}}{\partial z}\right|_h = h^{II}a^{II} + \frac{\mu^{II}W}{K_z}, \quad \left.\frac{\partial\Phi^{II}}{\partial x_j}\right|_h = \frac{\partial\Phi^{II}(h)}{\partial x_j} - a^{II}h^{II}\frac{\partial h}{\partial x_j} - \frac{\mu^{II}W}{K_z}\frac{\partial h}{\partial x_j}.$$

Then from (3.2c)

$$S^I = \frac{\partial\Phi^I(h)}{\partial x_j}\frac{\partial h}{\partial x_j} + a^I h\left(\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_j} + \frac{K_z}{K_x}\right),$$
$$S^{II} = \frac{\partial\Phi^{II}(h)}{\partial x_j}\frac{\partial h}{\partial x_j} - \left(a^{II}h^{II} + \frac{\mu^{II}W}{K_z}\right)\left(\frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_j} + \frac{K_z}{K_x}\right).$$

Finally, using (3.2b), we obtain

$$\left.\begin{aligned} a^I &= \left(\frac{\phi\mu^I}{K_x}\frac{\partial h}{\partial t} - \frac{\partial\Phi^I(h)}{\partial x_j}\frac{\partial h}{\partial x_j}\right)\left(h\left[\frac{K_z}{K_x} + \frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_j}\right]\right)^{-1}, \\ a^{II} &= -\left(\frac{\phi\mu^{II}}{K_x}\frac{\partial h}{\partial t} - \frac{\partial\Phi^{II}(h)}{\partial x_j}\frac{\partial h}{\partial x_j}\right)\left(h^{II}\left[\frac{K_z}{K_x} + \frac{\partial h}{\partial x_j}\frac{\partial h}{\partial x_j}\right]\right)^{-1} - \frac{\mu^{II}W}{K_z h^{II}}. \end{aligned}\right\} \tag{3.5}$$

We can now derive two additional formulae which relate functions R^I and R^{II} to other functions of the system (3.3). They are obtained from (3.4):

$$\overline{\Phi}^I = \Phi^I(h) + \frac{a^I h^2}{3}, \quad \overline{\Phi}^{II} = \Phi^{II}(h) + \frac{a^{II}(h^{II})^2}{3} + \frac{\mu^{II}h^{II}W}{2K_z}.$$

Taking into account (3.5), relation (3.1) then yields

$$\left.\begin{aligned} R^I &= \left(\frac{\phi\mu^I}{K_x}\frac{\partial h}{\partial t} - \frac{\partial\Phi^I(h)}{\partial x_j}\frac{\partial h}{\partial x_j}\right)\frac{h^2}{3\left[\dfrac{K_z}{K_x} + \dfrac{\partial h}{\partial x_j}\dfrac{\partial h}{\partial x_j}\right]}, \\ R^{II} &= -\left(\frac{\phi\mu^{II}}{K_x}\frac{\partial h}{\partial t} - \frac{\partial\Phi^{II}(h)}{\partial x_j}\frac{\partial h}{\partial x_j}\right)\frac{(h^{II})^2}{3\left[\dfrac{K_z}{K_x} + \dfrac{\partial h}{\partial x_j}\dfrac{\partial h}{\partial x_j}\right]} + \frac{\mu^{II}(h^{II})^2W}{6K_z}. \end{aligned}\right\} \tag{3.6}$$

The system of six equations (3.3) and (3.6) is now closed.

4. Final form of the model

4.1. *Dimensionless closed form of the averaged equations*

Let us introduce the dimensionless functions

$$\varphi(y_1, y_2, \tau) \equiv \frac{h}{h_0}, \quad \psi(y_1, y_2, \tau) \equiv \frac{h^{II}}{h_0^{II}}, \quad \tau \equiv \frac{t}{t_*}, \quad y_i \equiv \frac{x_i}{L},$$

$$\pi \equiv \frac{\Phi^I(h)}{\Delta P}, \quad r^I \equiv \frac{R^I}{h_0 \Delta P}, \quad r^{II} \equiv \frac{R^{II}}{h_0^{II} \Delta P}$$

After inserting (3.6) into (3.3) we obtain

$$\left.\begin{aligned}
&\mathscr{L}^I r^I + \varphi \mathscr{L}^I \pi = \frac{\tau_{gr}}{\omega}\frac{\partial \varphi}{\partial \tau} - \nabla\pi\cdot\nabla\varphi,\\
&-\mathscr{L}^{II} r^{II} - \psi \mathscr{L}^{II}\pi + \frac{\gamma}{\omega}\psi\mathscr{L}^{II}\varphi = \frac{\tau_{gr}\lambda}{\omega\overline{\mu}}\frac{\partial\varphi}{\partial\tau} - \lambda\nabla\pi\cdot\nabla\varphi + \frac{\gamma\lambda}{\omega}(\nabla\varphi)^2 - \alpha w(y,\tau),\\
&r^I = \frac{\varphi^2\varepsilon}{(1+\varepsilon(\nabla\varphi)^2)}\left[\frac{\tau_{gr}}{\omega}\frac{\partial\varphi}{\partial\tau} - \nabla\pi\cdot\nabla\varphi\right],\\
&r^{II} = \frac{\psi^2\varepsilon}{\lambda(1+\varepsilon(\nabla\varphi)^2)}\left[-\frac{\tau_{gr}}{\omega\overline{\mu}}\frac{\partial\varphi}{\partial\tau} + \nabla\pi\cdot\nabla\varphi - \frac{\gamma}{\omega}(\nabla\varphi)^2\right] + \frac{\varepsilon\alpha\psi^2 w(y,\tau)}{2\lambda^2},\\
&\psi = -\lambda\varphi + \lambda + 1,
\end{aligned}\right\} \tag{4.1}$$

where the new variables and operators are denoted

$$\mathscr{L}^I \equiv \Delta - \tau_{el}\frac{\partial}{\partial\tau}, \quad \mathscr{L}^{II} \equiv \Delta - \frac{\tau_{el}}{\overline{\mu}}\frac{\partial}{\partial\tau} \tag{4.2}$$

the symbol Δ denotes Laplace's operator written using variable y, $\nabla \equiv \text{grad}_y$.

The following set of dimensionless parameters defines the process:

$$\left.\begin{aligned}
&\omega \equiv \frac{\Delta P}{\varrho^I g h_0}, \quad \varepsilon \equiv \left(\frac{h_0}{L}\right)^2 \frac{K_x}{3K_z}, \quad \alpha \equiv \frac{\langle W\rangle \phi L}{\left(\dfrac{K_x \Delta P}{\mu^{II} L}\right) h_0^{II}}, \quad \gamma \equiv \frac{\varrho^I - \varrho^{II}}{\varrho^I},\\
&\overline{\mu} \equiv \frac{\mu^I}{\mu^{II}}, \quad \lambda \equiv \frac{h_0}{h_0^{II}}, \quad \tau_{gr} \equiv \frac{t_{gr}}{t_*}, \quad \tau_{el} \equiv \frac{t_{el}}{t_*}, \quad w(y) \equiv \frac{W(y)}{\langle W\rangle}.
\end{aligned}\right\} \tag{4.3}$$

Parameter ε is the ratio between the vertical and horizontal scales factorized by the permeability ratio. For the case examined here of a thin layer, ε is always small if the anisotropy of the medium is not too high. This is the basic assumption of the paper.

The limitation on the degree of anisotropy of the medium follows from the definition of parameter ε:

$$\frac{K_z}{K_x} = O\left(\frac{1}{\varepsilon}\left[\frac{h_0}{L}\right]^2\right).$$

In particular, if $h_0/L \sim \varepsilon^2$ (which corresponds to the assumption of a thin stratum), then $K_z/K_x \sim \varepsilon$.

Parameter ω reflects the ratio of the characteristic horizontal flow velocity to the characteristic vertical velocity. Parameter α is the ratio between the mean flow rate across the top boundary and the characteristic horizontal flow rate. Parameter γ is the relative difference between the fluid densities: it is positive when the upper fluid is lighter. Parameter $\overline{\mu}$ is the ratio of fluid viscosities: it is greater than 1 when the upper fluid has a low viscosity. $\langle W \rangle$ is the mean flow velocity across the top boundary, averaged over the overall boundary area. Velocity w is positive when the crossflow is directed upwards. Parameter λ is of order 1.

Three characteristic times define the system:

$$t_{el} = \frac{\mu^I L^2 \beta_* \phi}{K_x}, \quad t_{gr} = \frac{\mu^I \phi L^2}{K_x \varrho^I g h_0}, \quad t_{\Delta P} = \frac{\mu^I \phi L^2}{K_x \Delta P},$$

where t_{el} defines the propagation time of an elastic disturbance over the scale L; t_{gr} is the overall extraction time of the fluid from the medium through a lateral section due to gravitational drainage; $t_{\Delta P}$ is the overall extraction time of the fluid from the medium due to the fall in lateral pressure, ΔP.

Amongst these three times, the third, $t_{\Delta P}$, plays a special role: both the gravitational and the elastic times are intrinsic parameters of the system, whereas time $t_{\Delta P}$ is defined using a value ΔP introduced through the boundary-value conditions (see § 2.5). Hence, time $t_{\Delta P}$ may be considered as external.

Time t_* is selected from two intrinsic times:

$$t_* = \max\{t_{el}, t_{gr}\}. \tag{4.4}$$

4.2. *Asymptotic equations as* $\varepsilon \to 0$

In this paper we examine the asymptotic behaviour of system (3.5) when the thickness of the stratum is small. More strictly, we assume that the lesser values of the second order, $O(\varepsilon^2)$, can be neglected.

We can first exclude the terms r^i from (4.1) from which they may be obtained in the first approximation:

$$\left.\begin{aligned}
&\frac{\tau_{gr}}{\omega}\frac{\partial \varphi}{\partial \tau} - \nabla\pi\cdot\nabla\varphi - \varphi\Delta\pi + \tau_{el}\varphi\frac{\partial \pi}{\partial \tau} = \varepsilon\mathscr{L}^I\left(\varphi^2\left[\frac{\tau_{gr}}{\omega}\frac{\partial \varphi}{\partial \tau} - \nabla\pi\cdot\nabla\varphi\right]\right) + O(\varepsilon^2),\\
&\lambda\left(\frac{\tau_{gr}}{\omega\overline{\mu}}\frac{\partial \varphi}{\partial \tau} - \nabla\pi\cdot\nabla\varphi + \frac{\gamma}{\omega}(\nabla\varphi)^2\right) - \frac{\tau_{el}\psi}{\overline{\mu}}\left(\frac{\partial \pi}{\partial \tau} - \frac{\gamma}{\omega}\frac{\partial \varphi}{\partial \tau}\right)\\
&\qquad +\psi\Delta\pi - \frac{\gamma}{\omega}\psi\Delta\varphi - \alpha w(y,\tau)\\
&\qquad = \varepsilon\mathscr{L}^{II}\left(\frac{\psi^2}{\lambda}\left[\frac{\tau_{gr}}{\omega\overline{\mu}}\frac{\partial \varphi}{\partial \tau} - \nabla\pi\cdot\nabla\varphi + \frac{\gamma}{\omega}(\nabla\varphi)^2\right] - \frac{\alpha\psi^2 w(y,\tau)}{2\lambda^2}\right) + O(\varepsilon^2),\\
&\psi = -\lambda\varphi + \lambda + 1.
\end{aligned}\right\} \tag{4.5}$$

The first two equations can be simplified to the following form:

$$\left.\begin{aligned}
&\frac{\tau_{gr}}{\omega}\frac{\partial \varphi}{\partial \tau} - \operatorname{div}(\varphi\operatorname{grad}\pi) + \tau_{el}\varphi\frac{\partial \pi}{\partial \tau} = \varepsilon\mathscr{L}^I\left(\varphi^3\Delta\pi - \tau_{el}\varphi^3\frac{\partial \pi}{\partial \tau}\right),\\
&\frac{\tau_{gr}}{\omega\overline{\mu}}\frac{\partial \varphi}{\partial \tau} + \operatorname{div}(\psi\operatorname{grad}\pi) - \frac{\gamma}{\omega}\operatorname{div}(\psi\operatorname{grad}\pi) - \frac{\tau_{el}\psi}{\overline{\mu}}\left(\frac{\partial \pi}{\partial \tau} - \frac{\gamma}{\omega}\frac{\partial \varphi}{\partial \tau}\right) - \alpha w\\
&\qquad = \varepsilon\mathscr{L}^{II}\left(\frac{\psi^2}{\lambda^2}\left[\frac{\gamma}{\omega}\psi\Delta\varphi - \psi\Delta\pi + \alpha w + \frac{\tau_{el}\psi}{\overline{\mu}}\left(\frac{\partial \pi}{\partial \tau} - \frac{\gamma}{\omega}\frac{\partial \varphi}{\partial \tau}\right)\right] - \frac{\alpha\psi^2 w}{2\lambda^2}\right),
\end{aligned}\right\}$$

with the order of the remainder equal to $O(\varepsilon^2)$.

This system may be represented in an equivalent form, by eliminating the second derivative with respect to π in one equation:

$$\left.\begin{aligned}
&\frac{\tau_{gr}}{\omega}\left(\frac{\psi}{\varphi}+\frac{\lambda}{\overline{\mu}}\right)\frac{\partial\varphi}{\partial\tau}-\frac{(\lambda+1)}{\varphi}\nabla\pi\cdot\nabla\varphi-\frac{\gamma}{\omega}\mathrm{div}\,(\psi\mathrm{grad}\varphi)-\alpha w\\
&\quad+\frac{\tau_{el}\psi}{\overline{\mu}}\left((\overline{\mu}-1)\frac{\partial\pi}{\partial\tau}+\frac{\gamma}{\omega}\frac{\partial\varphi}{\partial\tau}\right)\\
&\quad=\varepsilon\frac{\psi}{\varphi}\mathscr{L}^{I}\left(\varphi^{3}\Delta\pi-\tau_{el}\varphi^{3}\frac{\partial\pi}{\partial\tau}\right)\\
&\quad+\varepsilon\mathscr{L}^{II}\left(\frac{\psi^{2}}{\lambda^{2}}\left[\frac{\gamma}{\omega}\psi\Delta\varphi-\psi\Delta\pi+\frac{\alpha w}{2}+\frac{\tau_{el}\psi}{\overline{\mu}}\left(\frac{\partial\pi}{\partial\tau}-\frac{\gamma}{\omega}\frac{\partial\varphi}{\partial\tau}\right)\right]\right),\\
&\tau_{el}\varphi\frac{\partial\pi}{\partial\tau}-\mathrm{div}(\varphi\mathrm{grad}\pi)+\frac{\tau_{gr}}{\omega}\frac{\partial\varphi}{\partial\tau}=\varepsilon\mathscr{L}^{I}\left(\varphi^{3}\Delta\pi-\tau_{el}\varphi^{3}\frac{\partial\pi}{\partial\tau}\right).
\end{aligned}\right\}\tag{4.6}$$

The natural limitation on the variation of the solution to model (4.6) is

$$0\leqslant\varphi\leqslant\frac{1+\lambda}{\lambda},\tag{4.7}$$

i.e. an interface cannot penetrate across the lower and the upper stratum boundaries.

4.3. *Relation for flow rates and averaged pressures*

To proceed further, the relation for flow rates averaged over the layer thickness will be required in order to set boundary conditions. Let us examine a cylindrical surface $\mathscr{F}$ orthogonal to the plane (y_1, y_2) and intersecting the top and the bottom of the domain Ω as depicted in figure 1 (we will keep the same notation $\mathscr{F}$ for the dimensionless variable y). This may be the surface of a well. Let $\mathscr{G}$ be a closed plate line, which results from the intersection of surface $\mathscr{F}$ with any orthogonal horizontal plane.

The volume flow rate of the upper and the lower liquids across the interface $\mathscr{F}$ is defined as

$$Q^{I}\equiv\int_{\mathscr{G}}\int_{0}^{h}V_{n}\,\mathrm{d}z\,\mathrm{d}\mathscr{G}=\frac{h_{0}L}{t_{*}}\int_{\mathscr{G}}q^{I}\,\mathrm{d}\mathscr{G},\quad Q^{II}\equiv\int_{\mathscr{G}}\int_{h}^{H}V_{n}\,\mathrm{d}z\,\mathrm{d}\mathscr{G}=\frac{h_{0}^{II}L}{t_{*}}\int_{\mathscr{G}}q^{II}\,\mathrm{d}\mathscr{G},\tag{4.8}$$

where V_n is the component of flow velocity normal to $\mathscr{F}$, and q^i are the dimensionless densities of the flow rates across $\mathscr{F}$ defined as

$$q^{I}\equiv\frac{t_{*}}{h_{0}L}\int_{0}^{h}V_{n}^{I}\,\mathrm{d}z,\quad q^{II}\equiv\frac{t_{*}}{h_{0}^{II}L}\int_{h}^{H}V_{n}^{II}\,\mathrm{d}z.$$

The averaged velocity is related to functions R^i and Φ^i as

$$\int_{0}^{h}V_{n}^{I}\,\mathrm{d}z=-\frac{K_{x}}{\mu^{I}}\int_{0}^{h}\frac{\partial\Phi^{I}}{\partial n}\,\mathrm{d}z=-\frac{K_{x}}{\mu^{I}}\left(\frac{\partial R^{I}}{\partial n}+h\frac{\partial\Phi^{I}(h)}{\partial n}\right),\ \text{etc.}$$

So for dimensionless fluxes it is easy to obtain the following relations in terms of variables φ, ψ, π:

$$\left.\begin{aligned} q^I &= -\frac{\omega\phi}{\tau_{gr}}\left\{\varphi\frac{\partial\pi}{\partial n}+\varepsilon\frac{\partial}{\partial n}\left(\frac{\tau_{gr}}{\omega}\varphi^2\frac{\partial\varphi}{\partial\tau}-\varphi^2\nabla\pi\cdot\nabla\varphi\right)\right\},\\ q^{II} &= -\frac{\overline{\mu}\omega\phi}{\tau_{gr}}\left\{\psi\frac{\partial\pi}{\partial n}-\frac{\gamma}{\omega}\psi\frac{\partial\varphi}{\partial n}+\frac{\varepsilon}{\lambda}\frac{\partial}{\partial n}\right.\\ &\quad\times\left.\left(\psi^2\nabla\pi\cdot\nabla\varphi-\frac{\tau_{gr}}{\omega\overline{\mu}}\psi^2\frac{\partial\varphi}{\partial\tau}-\frac{\gamma}{\omega}\psi^2(\nabla\varphi)^2+\frac{\alpha w(y,\tau)\psi^2}{2\lambda}\right)\right\}\end{aligned}\right\} \tag{4.9}$$

where $\partial/\partial n$ means derivatives along the normal direction to surface $\mathscr{F}$.

The averaged velocity potentials, $\bar{\pi}^i \equiv \overline{\Phi}^i/\Delta P$, can be determined in terms of r^i as

$$\overline{\pi}^I = \pi + \frac{r^I}{\varphi}, \qquad \overline{\pi}^{II} = \pi^{II} + \frac{r^{II}}{\psi}, \quad \text{with } \pi^{II} = \pi - \frac{\gamma}{\omega}\varphi.$$

Hence, taking into account two last equations in (4.1), we obtain

$$\left.\begin{aligned} \overline{\pi}^I &= \pi + \varepsilon\varphi\left(\frac{\tau_{gr}}{\omega}\frac{\partial\varphi}{\partial\tau}-\nabla\pi\cdot\nabla\varphi\right),\\ \overline{\pi}^{II} &= \pi - \frac{\gamma}{\omega}\varphi - \frac{\varepsilon}{\lambda}\psi\left(\frac{\tau_{gr}}{\omega\overline{\mu}}\frac{\partial\varphi}{\partial\tau}-\nabla\pi\cdot\nabla\varphi+\frac{\gamma}{\omega}(\nabla\varphi)^2\right).\end{aligned}\right\} \tag{4.10}$$

It can be seen that the difference between the averaged velocity potential and the potential at the interface disappears as $\varepsilon \to 0$. This means that the pressure distribution along z is practically hydrostatic, and that the vertical flow velocity is close to zero.

5. Gravity waves

5.1. *General nonlinear equations for gravity waves*

Let us examine the case where the elastic wave spreading time is very small compared to gravitational time. The third time $t_{\Delta P}$ can be of any length.

In this case, fluid/medium compressibility does not play any role, and therefore non-stationarity is introduced by gravitational phenomena only. Due to this, we call the solutions obtained 'gravity waves'.

The characteristic time scale t_* should be chosen as equal to t_{gr}, according to (4.4). Then $\tau_{gr} = 1$, $\tau_{el} \ll 1$.

We obtain from (4.6)

$$\left.\begin{aligned} &\frac{1}{\omega}\left(\frac{\psi}{\varphi}+\frac{\lambda}{\overline{\mu}}\right)\frac{\partial\varphi}{\partial\tau}-\frac{(\lambda+1)}{\varphi}\nabla\pi\cdot\nabla\varphi\\ &\quad=\frac{\gamma}{\omega}\mathrm{div}(\psi\,\mathrm{grad}\varphi)+\alpha w(y)+\varepsilon\frac{\psi}{\varphi}\Delta(\varphi^3\Delta\pi)\\ &\quad+\varepsilon\Delta\left(\frac{\psi^3}{\lambda^2}\left[\frac{\gamma}{\omega}\psi\Delta\varphi-\psi\Delta\pi+\frac{\alpha w}{2}\right]\right),\\ &-\mathrm{div}(\varphi\,\mathrm{grad}\pi)=-\frac{1}{\omega}\frac{\partial\varphi}{\partial\tau}+\varepsilon\Delta(\varphi^3\Delta\pi).\end{aligned}\right\} \tag{5.1}$$

The omitted terms are of $O(\varepsilon^2)$.

Due to the appearance of a third-order mixed derivative, it is difficult to determine the system type. It can be analysed, however.

5.2. *A particular case: free-boundary single-fluid flow*

For single-phase flow of the lower liquid with a free boundary φ (which describes in particular groundwater flow in contact with air in an unconfined aquifer), a generalization of the classical nonlinear Boussinesq equation can be deduced from (5.1). Let us assume the upper fluid has zero density and viscosity, i.e. $\gamma \to 1$ and $\overline{\mu} \to \infty$. The pressure at the free surface is constant and equal to atmospheric pressure. Hence $P^{II} = P_{at} = \text{const}$, $\pi = P_{at}/\Delta P + \varphi/\omega$, $\nabla\pi = (1/\omega)\nabla\varphi$. Thus, system (5.1) yields a single equation in the first approximation:

$$\frac{\partial\varphi}{\partial\tau} = \operatorname{div}(\varphi \operatorname{grad}\varphi) - \varepsilon\Delta(\varphi^3\Delta\varphi). \tag{5.2}$$

Exactly the same equation has been derived in Dagan (1967), Parlange *et al.* (1984) and Liu & Wen (1997) using the asymptotic expansion method, instead of integration along z.

In the zero approximation with respect to ε, the Boussinesq equation is obtained:

$$\frac{\partial\varphi}{\partial\tau} = \operatorname{div}(\varphi \operatorname{grad}\varphi). \tag{5.3}$$

This is the classical model of groundwater flow theory in an unconfined aquifer.

5.3. *Regimes of gravity flow*

The parametric analysis of the first equation in (5.1) enables us to formulate a classification of flow regimes.

(i) *A nonlinear diffusion regime* is observed when $\omega \to 0$, i.e. the horizontal flow velocity is low. This case describes the flow caused essentially by gravity. It concerns the free spreading of one liquid over another, or the flow of a stratified liquid towards a well under weak pressure gradients. In this case system (5.1) can be transformed into a single equation:

$$\left(\frac{\psi}{\varphi(\psi)} + \frac{\lambda}{\overline{\mu}}\right)\frac{\partial\varphi}{\partial\tau} = \gamma \operatorname{div}(\psi(\varphi)\operatorname{grad}\varphi) + \omega\alpha w(y,\tau). \tag{5.4}$$

The last term is significant if α is high. Note that (5.4) can be rewritten with respect to function ψ:

$$\left(\frac{\psi}{\varphi(\psi)} + \frac{\lambda}{\overline{\mu}}\right)\frac{\partial\psi}{\partial\tau} = \gamma \operatorname{div}(\psi \operatorname{grad}\psi) - \omega\alpha w(y,\tau). \tag{5.5}$$

(ii) *A convection–diffusion regime* is observed when $\omega \sim 1$. Equations (5.1) then take the form

$$\left.\begin{aligned} &\frac{1}{\omega}\left(\frac{\psi}{\varphi} + \frac{\lambda}{\overline{\mu}}\right)\frac{\partial\varphi}{\partial\tau} - \frac{(\lambda+1)}{\varphi}\nabla\pi\cdot\nabla\varphi = \frac{\gamma}{\omega}\operatorname{div}(\psi\operatorname{grad}\varphi) + \alpha w(y,\tau), \\ &\operatorname{div}(\varphi\operatorname{grad}\pi) = \frac{1}{\omega}\frac{\partial\varphi}{\partial\tau}. \end{aligned}\right\} \tag{5.6}$$

(iii) *A convection regime under weak dissipation* can be reached when $\omega \sim \varepsilon^{-1} \to \infty$, i.e. horizontal flow velocity is very high. In this case, all the second and third derivatives are grouped under a single dissipation term, which is of $O(\varepsilon + \omega^{-1})$. Although this term is small, it must not be entirely neglected, as it produces some dissipation phenomena in spatial boundary layers. This can be observed in the vicinity of the interface.

Then (5.1) take the form

$$\left.\begin{aligned} &\frac{1}{\omega}\left(\frac{\psi}{\varphi}+\frac{\lambda}{\bar{\mu}}\right)\frac{\partial\varphi}{\partial\tau}-\frac{(\lambda+1)}{\varphi}\nabla\pi\cdot\nabla\varphi \\ &\qquad=\alpha w(y)+\varepsilon\left\{\frac{\gamma}{\varepsilon\omega}\mathrm{div}\,(\psi\mathrm{grad}\varphi)+\frac{\psi}{\varphi}\Delta(\varphi^3\Delta\pi)-\frac{1}{\lambda^2}\Delta(\psi^3\Delta\pi-\alpha w\psi^2)\right\}, \\ &\mathrm{div}(\varphi\mathrm{grad}\pi)=\varepsilon\left\{\frac{1}{\varepsilon\omega}\frac{\partial\varphi}{\partial\tau}-\Delta(\varphi^3\Delta\pi)\right\}. \end{aligned}\right\} \quad (5.7)$$

A small parameter before the time derivative represents the appearance of a time boundary layer phenomenon, outside which the system behaviour becomes stationary. To study the development of transient phenomena it is sufficient to rescale the time by introducing the new time $\theta=\omega\tau$.

5.4. *Rayleigh–Taylor instability in dynamics*

One of the ways to study interface behaviour is via a stability analysis. Two basic types of instability usually appear in conjunction with the shear flow of a stratified liquid: the Rayleigh–Taylor instability caused by gravity forces (Chandrasekhar 1961; Inogamov, Demianov & Son 1999), and the Kelvin–Helmholtz instability caused by inertia effects (Chandrasekhar 1961). In a Hele-Shaw cell, the Rayleigh–Taylor instability has been examined in Maxworthy (1987), the Kelvin–Helmholtz instability in Yih (1967), Thorpe (1969), and Gondret & Rabaud (1989).

Within the framework of assuming the neglect of inertia forces, no mechanisms exist enabling the Kelvin–Helmholtz instability to occur. On the other hand, within the framework of assuming the neglect of surface tension, no mechanisms exist enabling the develoment of Rayleigh–Taylor instability to be counteracted. Hence the model suggested is not suited to stability analysis.

However, it is easy to show that this model sharply changes its properties at the instability threshold, as it is sensitive to Rayleigh–Taylor instability. Indeed, equations (5.4) and (5.6) show that the diffusion term, which is defined as

$$D(\varphi)=\frac{\gamma}{\omega}\psi, \quad (5.8)$$

can change sign depending on the sign of the gravitational parameter γ. In particular, when

$$\gamma<0 \quad (5.9)$$

(the upper fluid is heavier) the diffusion term becomes negative. Hence, we obtain the anti-diffusion equation. It is well-known that the Cauchy problem for an anti-diffusion equation is unstable. We can thus conclude that the classical Rayleigh–Taylor gravitational instability is associated with the negative sign of the diffusion parameter in the equation for φ.

In the case of a weak or moderate horizontal flow represented by (5.4) and (5.6) (the diffusion regime and the convection–diffusion regime), the problem of gravitational instability is rather trivial and has the same solution as under the static condition: the flow is always unstable if (5.9) holds. Thus the flow dynamic does not influence system stability.

Physically the appearance of the anti-diffusion effect as a model of gravitational instability can be explained in the following manner. In the theory of dissipative structures, the anti-diffusion equation arises as a model of an unstable homogeneous multicomponent mixture which tends (over time) be split into stable, single-component sub-domains, thus forming a highly heterogeneous system (this is opposite to the diffusion effect which tends to homogenize the mixture). The same tendency is inherent in a gravitationally unstable system, which tends to be split into two stable, single-phase layers by passing through an unstable mixture state.

Note that equations with variations in the parabolicity direction were considered in Plotnikov (1993), as a model of hysteresis phenomena in the theory of phase transitions. However, this theory cannot be applied to (5.1), as the structure of nonlinearity was very different. On the other hand, certain methods of regularization of equations such as (5.1) were examined. According to Plotnikov (1993), the best regularization can be attained by adding the term $\Delta\partial\varphi/\partial\tau$. Then, the full equation (5.1) is expected to regularize unstable solutions.

The conclusion of this section provides us with a probable tool for analysing gravitational instability in complicated cases. According to our hypothesis, to check whether the system is gravitationally unstable, it is sufficient to check the sign of the formal diffusion parameter in the equation describing the interface dynamics.

6. The spreading of one liquid over another: degenerating cases

Let us examine the stable case only: $\gamma > 0$. One of the basic properties of model (5.4) is the existence of singular points when $\psi \to 0$ and $\varphi \to 0$. These two cases mean a degeneration of the two-phase system: in the first case the upper fluid disappears; in the second case the lower fluid is completely replaced by the upper one. An analysis of these singularities can be performed based on the problem of contact between two fluids.

6.1. *Problem of contact between two liquids*

Let us examine the one-dimensional problem of contact between two liquids in the domain $(-\infty < y < +\infty)$, assuming that the initial interface is vertical, with $y \equiv y_1$ being the Cartesian coordinate. The right-hand half-space is filled by the heavier fluid, while the left-hand half contains the lighter fluid over an initial thin layer of the heavier fluid φ_0 in height. In particular, φ_0 can be equal to 0. This problem is described by the following one-dimensional initial problem for function $\varphi(y, \tau)$:

$$\varphi|_{\tau=0} = \varphi_0 H(y) + (1+\lambda)(1 - H(y)), \tag{6.1}$$

where $H(y)$ is the Heaviside function.

Due to the density difference, the vertical interface cannot remain in the steady state and begins to deform similar to a diffusion wave. The heavier liquid flows under the lighter liquid by forming a singularity $\varphi \to 0$ if $\varphi_0 = 0$, while the lighter liquid tends to form a layer covering the heavier liquid, by determining the singularity $\psi \to 0$.

6.2. *Degeneration* $\psi \to 0$

When $\psi \to 0$ equation (5.5) has the following asymptotic form:

$$\frac{\lambda}{\bar{\mu}}\frac{\partial\psi}{\partial\tau} = \gamma \mathrm{div}(\psi \mathrm{grad}\psi). \tag{6.2}$$

This is a well-known nonlinear diffusion equation studied in Barenblatt *et al.* (1990) and Samarski *et al.* (1995). In particular, this equation has a solution with a finite velocity of wave propagation. This situation is illustrated by the left-hand side of figure 3 below where $\xi = y/\sqrt{\gamma\tau}$. It is easy to show that, at point ξ_*, the incidence of

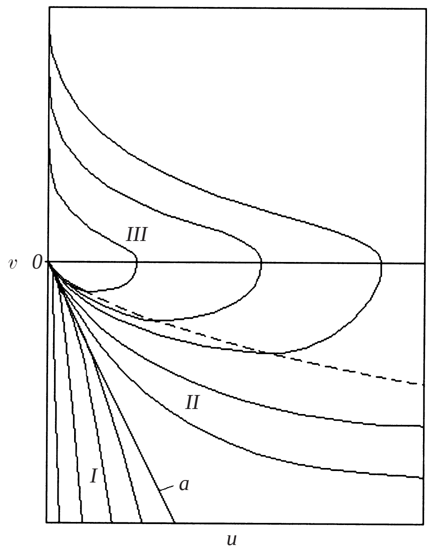


FIGURE 2. Phase portrait of the spreading equation.

the wave is equal to

$$\left.\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right|_{\xi\to\xi_*} = -\frac{\lambda\xi_*}{2\bar{\mu}}.$$

6.3. *Degeneration* $\varphi \to 0$

Let $\varphi_0 = 0$. When $\varphi{\to}0$ equation (5.4) has the following asymptotic form:

$$\frac{\partial\varphi}{\partial\tau} = \gamma\varphi\Delta\varphi \tag{6.3}$$

or the self-similar form: $-\xi\psi'/2 = \varphi(\varphi)''$, $\xi \equiv y/\sqrt{\gamma\tau}$.

By the replacement of variables $\varphi(\xi) = \xi^2 u(\eta)$, $\eta = \ln\xi$, the last equation may be transformed into the following:

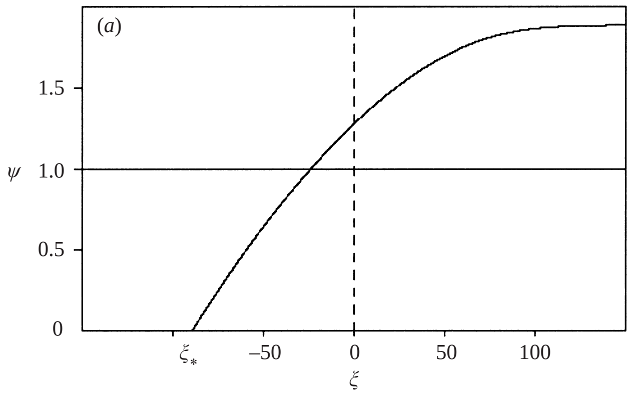
$$2u(u'' + 3u' + 2u) = -2u - u',$$

which can be reduced to the first-order equation in the usual way:

$$\frac{\mathrm{d}v}{\mathrm{d}u} = -\frac{2u + v + 6uv + 4u^2}{2uv}, \quad v \equiv \frac{\mathrm{d}u}{\mathrm{d}\eta}. \tag{6.4}$$

The phase portrait of this equation for $u \geqslant 0$ is shown in figure 2.

Three families of integral curves determine behaviour in the vicinity of the degeneration point η^* where $u \to 0$. Family *III* tends to zero with a positive derivative $v = \mathrm{d}u/\mathrm{d}\eta$, which increases toward infinity. This solution has no physical meaning, as the infinite derivative stands for an infinite flow rate. Families *II* and *I* have a negative derivative $\mathrm{d}u/\mathrm{d}\eta$, which tends to zero along the characteristic line $v = -2u$ (curve a in figure 2). So, any solution decreases as $u = C\exp(-2\eta)$, C being an arbitrary constant. This means however that, in terms of function $\varphi(\xi)$, any solution behaves as $\varphi(\xi) \sim C$. According to the boundary conditions, $C = \varphi_0$. Thus, a non-trivial function $\varphi(\xi)$ cannot intersect axis ξ wherever the point ξ^*, including $\xi^* \to \infty$. (A trivial function corresponds to $C = 0$.) Thus, the problem (6.1) has solutions if $\varphi_0 > 0$.



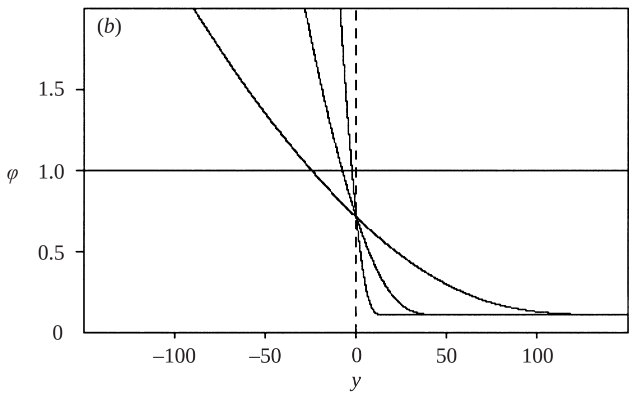


FIGURE 3. Solution of the contact problem.

This analysis shows that the heavier fluid penetrates under the lighter fluid if an initial layer of the heavier liquid already exists in the initial state.

6.4. *Self-similar solution*

Problem (6.1) for (5.5) has a similar solution in the form $\varphi = \varphi(\xi)$, $\xi = y/\sqrt{\gamma\tau}$ which is clearly the solution to the following boundary-value problem for the ODE:

$$-\frac{\xi}{2}\left(\frac{\psi}{\varphi(\psi)} + \frac{\lambda}{\overline{\mu}}\right)\psi' = (\psi(\psi)')', \quad \psi|_{\xi\to-\infty} = 0, \quad \psi|_{\xi\to+\infty} = 1 + \lambda. \qquad (6.5)$$

The inflow w is neglected.

The solution behaviour is shown in figure 3 in terms of functions $\psi(\xi)$, and $\varphi(y, \tau)$ for three instants of time. The parameters are $\varphi_0 = 0.15$; $\lambda = 1$; $\gamma = 0.1$; $\overline{\mu} = 1$.

7. Oil–water interface in an oil reservoir

The stable case of a rather low lateral flow velocity describes the behaviour of oil reservoirs quite well.

7.1. *Physical description of the system*

Let us examine the problem of oil extraction from a porous reservoir by a well. The lower layer is saturated by water (index I), the upper layer is oil saturated. The gravitational parameter γ is always positive, such that the system is gravitationally stable. The stratum is penetrated by a well. The well is assumed to be of multitube construction, such that each fluid can be extracted separately through its own tube. The engineering problem consists in knowing how to control the deformations of the oil–water interface in order to reduce the extraction of water. Such technology has been analysed, for instance, in Korotaev & Zakirov (1981). A similar problem arises in a gas–oil system. Here, indexes I and II designate oil and gas respectively. Let us examine the following problem of radial flow towards a single well located in the centre of a cylindrical porous domain of radius L, height H, porosity m and permeability K. Let the well be a vertical cylinder of radius R_w. Let $Q^i_{(i=I,II)}$ be the volume flow rate of extraction of the ith fluid from the well.

Let us examine the case of small external horizontal gradients: $\omega \ll 1$. We can then use the system of equations (5.8) corresponding to the asymptotics ω, $\varepsilon \to 0$. In practice, the pressure drop ΔP is in the order of 1 MPa, while the pressure of the gravity column $\varrho g h_0$ is in the order of 10 MPa if the reservoir thickness is 1000 m. Thus $\omega = 0.1$.

7.2. *Mathematical formulation*

For the flow rates across the well surface we can use general relations (4.8) and (4.9). Assuming the flow velocity at the well does not depend on the polar angle, we obtain

$$\int_{\mathscr{G}} q^i \mathrm{d}\mathscr{G} = 2\pi R_w q^i, \quad Q^I = \frac{2\pi R_w h_0 L}{t_{gr}} q^I \bigg|_{R_w}, \quad Q^{II} = \frac{2\pi R_w h_0^{II} L}{t_{gr}} q^{II} \bigg|_{R_w},$$

$$q^I = -\omega\phi\varphi\frac{\partial\pi}{\partial r}, \quad q^{II} = -\overline{\mu}\omega\phi\psi\left\{\frac{\partial\pi}{\partial r} - \frac{\gamma}{\omega}\frac{\partial\varphi}{\partial r}\right\},$$

where $r = R/L$, $r_w = R_w/L$.

Then we obtain the following boundary-value conditions:

$$r\varphi\frac{\partial\pi}{\partial r}\bigg|_{r=r_w} = \frac{Q^I t_{gr}}{2\pi\omega h_0 L^2\phi}, \quad r\psi\left\{\frac{\partial\pi}{\partial r} - \frac{\gamma}{\omega}\frac{\partial\varphi}{\partial r}\right\}_{r=r_w} = \frac{Q^{II} t_{gr}}{2\pi\omega h_0^{II} L^2 \phi\overline{\mu}}.$$

In this case the boundary values Q^i are given instead of ΔP used earlier, but they can be related via the Darcy law: $\Delta P = Q^I\mu^I/(2\pi K_x h_0)$. Then, by eliminating the term $\partial\pi/\partial n$ from the second condition, we obtain finally

$$\left\{\frac{\psi}{\varphi} - \frac{2\gamma}{\omega}r\psi\frac{\partial\varphi}{\partial r}\right\}_{r=r_w} = \frac{\lambda}{\overline{Q}\overline{\mu}}, \quad r\varphi\frac{\partial\pi}{\partial r}\bigg|_{r=r_w} = \frac{1}{2}, \quad \overline{Q} \equiv \frac{Q^I}{Q^{II}}. \tag{7.1}$$

After some simple transformations we obtain the boundary-value problem for $\varphi(r,\tau)$, which follows from (5.8):

$$\left.\begin{aligned}
&\left(\frac{\psi}{\varphi} + \frac{\lambda}{\overline{\mu}}\right)\frac{\partial\varphi}{\partial\tau} = \frac{\gamma}{r}\frac{\partial}{\partial r}\left(r\psi\frac{\partial\varphi}{\partial r}\right), \quad r \in (r_w, 1), \quad \tau > 0,\\
&\left\{\frac{\psi}{\varphi} - \frac{2\gamma}{\omega}r\psi\frac{\partial\varphi}{\partial r}\right\}_{r=r_w} = \frac{\lambda}{\overline{Q}\overline{\mu}},\\
&\varphi|_{r=1} = 1, \quad \varphi|_{\tau=0} = 1.
\end{aligned}\right\} \tag{7.2}$$

The conditions at $\tau = 0$ and at $r = 1$ mean that the interface was undisturbed in the initial state and remains undisturbed far from the well. The omitted terms in (7.2) are of $O(\varepsilon^2)$.

General limitation (4.7) remains true:

$$0 \leqslant \varphi \leqslant \frac{1+\lambda}{\lambda}. \tag{7.3}$$

7.3. *Critical regime of oil extraction*

System (7.2) is characterized by a constant critical flow rate ratio defined as

$$\overline{Q}_* = \frac{\lambda}{\overline{\mu}}. \tag{7.4}$$

If $\overline{Q} = \overline{Q}_*$ the system remains in the undisturbed state, or $\varphi \equiv 1$. This theorem is easy to prove. It can be seen from (7.2) that the derivative $\partial\varphi/\partial r$ at $r = r_w$ is equal to zero by identity when

$$\overline{Q}(\tau) = \frac{\varphi}{\psi}\bigg|_{r=r_w} \frac{\lambda}{\overline{\mu}}. \tag{7.5}$$

In this case problem (7.2) describes an undisturbed system. Its solution is unique and simple: $\varphi \equiv 1$.

According to the initial condition, $\varphi = 1$ at $\tau = 0$. Then $\psi|_{\tau=0} = 1$ also, according to the definition of ψ (see (5.7), for instance). Hence, at the initial instant, the ratio φ/ψ in (7.5) is equal to 1. Therefore the initial value of $\overline{Q}$ ensuring the undisturbed state for the interface is equal to the critical value (7.4). In other words, the critical value (7.4) ensures the initial, undisturbed state of the interface. Hence φ and ψ, as well as their ratio φ/ψ will maintain their initial values at all other times.

Depending on the value of $\overline{Q}$ with respect to the critical value (7.4), the derivative $\partial\varphi/\partial r$ at $r = r_w$ can be zero, positive, or negative. Therefore three modes of flow, depicted in figure 4, may be distinguished.

(1) *Fluid extraction under an undisturbed interface* is observed when

$$\overline{Q} = \overline{Q}_* \quad \text{or} \quad \frac{Q^I/h_0}{Q^{II}/h_0^{II}} = \frac{\mu^{II}}{\mu^I}.$$

The physical meaning of this equation is clear: for the interface to remain horizontal the fluid velocities must be inversely proportional to the fluid viscosities. This criterion means that the pressure gradients are equal in both fluids, as has been shown in Zeybek & Yortsos (1992).

(2) *Extraction under invasion of the upper fluid* is observed when

$$\overline{Q} > \overline{Q}_* \quad \text{or} \quad \frac{Q^I/h_0}{Q^{II}/h_0^{II}} > \frac{\mu^{II}}{\mu^I}.$$

(3) *Extraction under invasion of the lower fluid* is observed when

$$\overline{Q} < \overline{Q}_* \quad \text{or} \quad \frac{Q^I/h_0}{Q^{II}/h_0^{II}} < \frac{\mu^{II}}{\mu^I}.$$

7.4. *Asymptotic solution*

As parameter ω is small, the asymptotic method can be used. Let the solution of (7.2) be of the form

$$\varphi(y,\tau) = 1 + \omega\varphi_1 + \omega^2 \ldots \, .$$

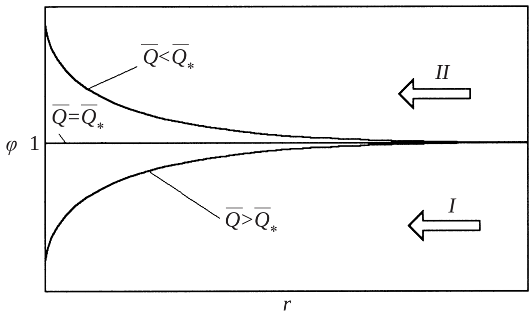


FIGURE 4. Three modes of the oil–water interface in the vicinity of a well.

For the first term, φ_1, it is easy to obtain the linear problem:

$$\left.\begin{aligned} &a\frac{\partial \varphi_1}{\partial \tau} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi_1}{\partial r}\right), \\ &r\frac{\partial \varphi_1}{\partial r}\bigg|_{r=r_w} = \frac{1}{2\gamma}\left[\frac{\lambda}{\overline{Q}\overline{\mu}} - 1\right], \\ &\varphi_1|_{r=1} = 0, \quad \varphi_1|_{\tau=0} = 0, \end{aligned}\right\} \tag{7.6}$$

where $a \equiv (1/\gamma)(1 + \lambda/\overline{\mu})$. The solution to this equation can be represented in an analytical manner. The most simple form is obtained for the self-similar solution, which corresponds to the case of an infinite reservoir ($L \to \infty$) and a very thin well ($R_w \to 0$). The solution is then: $\varphi_1 = \varphi_1(\xi)$, $\xi \equiv r/\sqrt{\tau}$. The problem takes the form

$$-\frac{a\xi}{2}\frac{\mathrm{d}\varphi_1}{\mathrm{d}\xi} = \frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\varphi_1}{\mathrm{d}\xi}\right), \quad 0 < \xi < \infty,$$

$$\varphi_1|_{\xi=\infty} = 0, \quad \xi\frac{\mathrm{d}\varphi_1}{\mathrm{d}\xi}\bigg|_{\xi=0} = \frac{1}{2\gamma}\left[\frac{\lambda}{\overline{Q}\overline{\mu}} - 1\right].$$

The solution has the following form:

$$\varphi_1 = \frac{1}{4\gamma}\left[\frac{\lambda}{\overline{Q}\overline{\mu}} - 1\right]\mathrm{Ei}\left(-\frac{a\xi^2}{4}\right), \quad \mathrm{Ei}(x) \equiv \int_{-\infty}^{x}\frac{\mathrm{e}^u}{u}\,\mathrm{d}u, \tag{7.7}$$

where Ei is the integral exponential function. Using its property, we obtain for $\xi \to 0$

$$\varphi = 1 + \frac{\omega}{4\gamma}\left[\frac{\lambda}{\overline{Q}\overline{\mu}} - 1\right]\left[\ln\frac{a\xi^2}{4} + Ce + \cdots\right] + O(\omega^2), \tag{7.8}$$

where $Ce = 0.5772\ldots$ is the Euler constant. This function is presented in figure 5 (curve b) and has exactly one of three forms shown in figure 4.

The comparison of an asymptotic (when $\omega \to 0$) solution (7.7) to the exact numerical solution of nonlinear problem (7.2) is presented in figure 5, for $\lambda = 1$, $\overline{\mu} = 1$, $\gamma = 1$, $\overline{Q} = 10$, $\omega = 0.01$. It is seen that the asymptotic solution is very close to being exact, excluding a narrow zone in the vicinity of point $r = 0$ where asymptotic function (7.8) is infinitely deformable. Indeed, the asymptotic solution describes a linearized form of problem (7.2) and, thus, cannot describe the singularities $\psi = 0$ and $\varphi = 0$ which were examined in §6, just as it cannot satisfy limitations (7.3). However, if the well radius is not zero, the solution to the asymptotic problem is bounded everywhere and gives a better description of the system behaviour in the vicinity of the well.

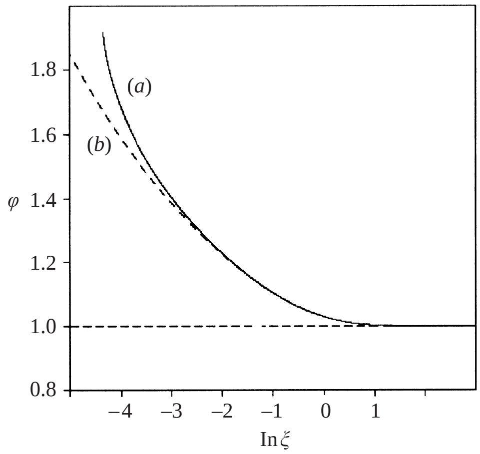


FIGURE 5. Comparison of the asymptotic (b) and the numerical nonlinear solution (a) for the problem of oil–water flow towards a well.

7.5. *Physical interpretation*

The critical mode of flow (7.4) means that the interface can be maintained in a dynamic undisturbed state if the mean velocity ratio of fluid extraction $V^i \equiv Q^i/h_i^0 = Q^i/\lambda$ is inversely proportional to the viscosity ratio. For the water–oil system at $\overline{\mu} \sim 0.1$–0.01 the oil velocity must be 10–100 times lower than that of water. If the water velocity is reduced below the critical value, so that $Q^{water}/Q^{oil} < Q_*$, then the well will be invaded by water. To prevent water invasion, the oil flow rate must be reduced below the critical value.

For the oil–gas system, characterized by $\overline{\mu} \sim 100$–1000, the interface remains undisturbed if the gas velocity is 100–1000 times greater than that of oil. If the gas velocity is reduced below the critical, the well will be invaded by gas.

The final conclusion concerns the stability of the critical regime. It is easy to see that solution (7.8) is always singular at $r \to 0$ or $\tau \to \infty$ when the flow rate ratio differs from the critical value. This means that the smallest disturbance of the critical regime leads to singular, infinite growth of the interface in the vicinity of the well. Thus, the critical regime is unstable.

8. Dominant elastic forces

Let us examine the case of a highly deformable medium and/or fluids, when the elastic wave propagation time is high compared to gravitational time. The time scale t_* should be chosen as t_{el}, according to (4.4). Then $\tau_* = 1$, $\tau_{gr} \ll 1$. Therefore, we obtain from (4.5) at $\varepsilon \to 0$

$$\varphi \frac{\partial \pi}{\partial \tau} = \operatorname{div}(\varphi \operatorname{grad} \pi), \tag{8.1a}$$

$$\frac{\psi}{\overline{\mu}} \frac{\partial \varphi}{\partial \tau} - \operatorname{div}(\psi \operatorname{grad} \varphi) = \frac{\omega}{\gamma} \left[\frac{\psi}{\overline{\mu}} \frac{\partial \pi}{\partial \tau} - \operatorname{div}(\psi \operatorname{grad} \pi) \right] + \frac{\alpha \omega w(y,\tau)}{\gamma}. \tag{8.1b}$$

Relations (4.9) for flow rates across a cylindrical surface $\mathscr{F}$ (figure 1) take the form

$$q^I = -\frac{\omega \phi}{\tau_{gr}} \left\{ \varphi \frac{\partial \pi}{\partial n} \right\}, \quad q^{II} = -\frac{\overline{\mu} \omega \phi}{\tau_{gr}} \left\{ \psi \frac{\partial \pi}{\partial n} - \frac{\gamma}{\omega} \psi \frac{\partial \varphi}{\partial n} \right\}. \tag{8.2}$$

Let us examine the same physical problem as in § 7 but within the framework of the model (8.1). Examining a low disturbance, which is always the case in practice, let us assume parameter ω to be small. Then function π can be excluded from (8.1). Let the surface $\mathscr{F}$ be a cylindrical well of dimensionless radius r_w as in 7.2. The top surface of the domain is impermeable ($\alpha = 0$). The problem of oil–water extraction under fixed flow rates Q^I and Q^{II} for each fluid hence takes the form

$$\left.\begin{aligned} &\frac{\psi}{\bar{\mu}}\frac{\partial \varphi}{\partial \tau} = \operatorname{div}(\psi \operatorname{grad}\varphi), \quad r \in (r_w, 1), \quad \tau > 0, \\ &\left\{\frac{\psi}{\varphi} - \frac{2\gamma}{\omega} r\psi \frac{\partial \varphi}{\partial r}\right\}_{r=r_w} = \frac{\lambda}{\overline{Q}\bar{\mu}}, \\ &\varphi|_{r=1} = 1, \quad \varphi|_{\tau=0} = 1, \end{aligned}\right\} \tag{8.3}$$

where as noted earlier $\Delta P = Q^I \mu^I / 2\pi K_x h_0$.

The last differential equation is linear with respect to the new function $f(y, \tau)$ defined as $\mathrm{d}f = \psi \mathrm{d}\varphi$. But the boundary-value conditions remain nonlinear.

The asymptotic solution at $\omega \to 0$ has the same form as (7.8) for a gravity-dominated mode, except for some differences in parameter magnitudes:

$$\varphi = 1 + \frac{\omega}{4\gamma}\left[\frac{\lambda}{\overline{Q}\bar{\mu}} - 1\right] \mathrm{Ei}\left(-\frac{\xi^2}{4\bar{\mu}}\right), \quad \xi = r/\sqrt{\tau}.$$

All conclusions about the critical flow rate ratio defined as (7.4) remain true.

9. Conclusion

A general model of interface deformation is obtained based on the shallow-water approach (the vertical size of the domain is much smaller than the horizontal). A single assumption has been used: a linear distribution for the vertical flow velocity both above and below the interface. The traditional assumption of low interface deformations has not been used, hence the model obtained also describes large deformations.

For the gravitational mode of flow (gravity drainage time is much greater than that of the elastic disturbance) three regimes have been detected. These differ in the ratio between vertical and lateral flow velocities. The gravitational phenomena are revealed in a nonlinear diffusion term, whilst lateral flow appears in the form of a convection term. Therefore in the case of weak lateral flow the model takes the form of a nonlinear diffusion equation, which can be considered as a generalization of the classical Boussinesq equation for water flow in contact with air in an unconfined aquifer. The case of a moderate lateral flow is described by the convection–diffusion equation, while the case of high lateral flow is modelled by a convection transport equation with small dissipative terms described by the second- and third-order derivatives.

Under a weak or moderate lateral flow, the apparent diffusion parameter becomes negative, leading to instability, when the upper fluid is heavier (this is the condition of Rayleigh–Taylor instability). Thus the Rayleigh–Taylor instability is shown to be associated with the transformation of a diffusion equation into an anti-diffusion equation. This result provides us with a tool to analyse gravitational instability in the more complicated case of stratified flow.

The stable case of flow under a weak lateral flow velocity (weak lateral disturbance) provides a good description of the oil–water or the gas–oil flow into a well in petroleum reservoirs. The boundary-value problem of radial flow into a single well in a cylindrical reservoir is examined assuming the flow rates at the wellbore of each fluid are known and are independent of one another. The analysis of the problem indicates the existence of three modes of well functioning depending on the flow rate ratio. When the ratio of the mean flow velocities is inversely proportional to the fluid viscosities, the interface remains undisturbed. Otherwise, the interface undergoes a very large and very rapid deformation in the vicinity of the well. Due to this, the critical mode of flow is unstable. The analytical solution to the problem is obtained using the disturbance method. These results give an estimation of the flow rate ratio into the oil reservoir required to prevent the well being invaded by water.

The second example which can be described by the suggested model is that of the free spreading of one liquid over another from the initial state of contact with a vertical interface. It is shown that the heavier liquid propagates into the reservoir by displacing the lighter liquid with an infinite velocity, if a thin initial layer of heavier liquid was already present under the lighter fluid. The propagation of the lighter fluid in the opposite direction has a finite velocity. This corresponds to a degeneration of the equations obtained.

The case where the gravitational drainage time is quicker than the elastic wave propagation time across the entire reservoir gives rise to another model of interface deformation. This is represented by a nonlinear diffusion equation and is formally close to the gravitational flow model under weak lateral pressure gradients. However, the model parameters are different. This case is analysed in the same way, using asymptotic methods.

Appendix

Let us demonstrate the validity of the assumption of a linear distribution of the vertical velocity along z.

The original equations (2.1) can be written in the following equivalent dimensionless form:

$$L^I f^I = -\frac{1}{3\varepsilon}\frac{\partial^2 f^I}{\partial z^2}, \quad L^{II} f^{II} = -\frac{1}{3\varepsilon}\frac{\partial^2 f^{II}}{\partial z^2},$$

where $f^i(y_1, y_2, z) \equiv \Phi^i/\Delta P$, operators L^i are defined in (4.2) and parameter ε is the dimensionless stratum thickness.

Since ε is small, the zero term of the asymptotic expansion for functions f^i does not depend on z:

$$f^i = f_0^i(y_1, y_2) + \varepsilon f_1^i(y_1, y_2, z) + \varepsilon^2 \ldots, \quad i = I, II.$$

Hence for the first approximation we obtain

$$\frac{\partial^2 f_1^i}{\partial z^2} = -3L^i f_0^i,$$

where the right-hand side does not depend on z. This is an ordinary differential equation with respect to function $f_1^i(z)$ which has a simple solution:

$$f_1^i = -\tfrac{3}{2}(L^i f_0^i)z^2 + C_1^i(y_1, y_2)z + C_2^i(y_1, y_2),$$

C_1^i and C_2^i being the integration constants. Thus in the first approximation the pressure is quadratic while the vertical flow velocity $v_z^i = \partial f^i/\partial z$ is linear along z.

Note that vertical velocity can be considered as linear along z for a thin stratum only. For strata of greater thicknesses, other nonlinear distributions can be obtained. This explains the diversity of data which may be found in the literature.

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